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Relative equilibria in systems with configuration space isotropy

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Abstract

We present a framework for studying Lagrangian and Hamiltonian systems with symmetries, near points with configuration space isotropy. We use twisted parametrisations corresponding to phase space slices based at zero points of (co-)tangent fibres. Given a hyperregular Lagrangian, we find a Legendre transform in the twisted coordinates. For simple mechanical systems, we state necessary and sufficient conditions for the existence of relative equilibria in terms of an *augmented-amended potential*.

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1. Introduction

The study of symmetry in mechanical systems has a very long history, going back to the foundations of classical mechanics. Typical questions include: how can we exploit the symmetry to simplify the problem? To what degree can we separate rotational from vibrational motion? What are the simplest or generic dynamical behaviors of a symmetrical system? The simplest non-equilibrium solution of a symmetrical system is a *relative equilibrium*, which is a solution that moves only in a symmetry direction; an example is the steady spinning of a rigid body around one of its principal axes. Like equilibria in general, these solutions can be used as organizing centres for understanding more complex dynamics.

In recent decades, symmetry has received particular attention in the field of geometric mechanics. Some key achievements are: the theory of symplectic reduction (see for example [1,2,19,33,26]); the Marle–Guillemin–Sternberg normal form (the Hamiltonian slice theorem) [12,6]; and the energy-momentum method [27,13,10,22]. These and many related results have given a firm geometrical foundation to the study of symmetry of Hamiltonian systems on symplectic manifolds. Many important mechanical systems have phase space which are (co-)tangent bundles, with (co-)tangent lifted symmetries. The geometrical theory of symmetry specific to (co-)tangent bundles has also seen many recent advances (see for example [11,15,13,16,5,32,29]). However, our understanding of lifted symmetries with non-trivial isotropy is far from complete.

The present paper considers Lagrangian and Hamiltonian systems on (co-)tangent bundles, with lifted symmetries and configuration-space isotropy. We present a practical geometrical framework for studying such systems using degenerate parametrisations of neighbourhoods of phase space points with configuration-space isotropy. The parametrisations are defined in Sections 2 and 3; they are tubes (defined below) around zero points in the (co-)tangent fibres. We find a Legendre transform in the new coordinates (Section 4), which allows us to study relative equilibria of Lagrangian systems in a Hamiltonian context. Finally, we study simple mechanical systems (Section 6), giving a set of necessary and sufficient conditions for the existence of a relative equilibrium, expressed in terms of an *augmented-amended potential* which generalises both the augmented potential and amended potential familiar from systems with free symmetries (see [13]).

For the motivation of this paper we are indebted to a series of molecular physicists and chemists, going back at least to Watson [34]. Our results in Sections 2–4 provide a theoretical foundation for techniques they have used in particular examples. Our geometrical formulation of these techniques builds on the presentation in [9].

We begin by summarising relevant basic facts about Lie group symmetries, including Palais' slice theorem (see [1,3,7,4,14,26]). The starting point of our approach will be to apply the slice theorem in a configuration space Q. We will then use the resulting parametrisation to study the phase spaces TQ and T^*Q.

1.1. Basic definitions and notation

Let G be a Lie group, with Lie algebra $\mathfrak{g}$, and consider a smooth left action of G on a finite-dimensional manifold M, written $(g, q) \mapsto g \cdot q$. For every $\xi \in \mathfrak{g}$ and $z \in M$, the *infinitesimal action* of ξ on z is

$$\xi \cdot z = \left.\frac{\mathrm{d}}{\mathrm{d}t} \exp(t\xi) \cdot z\right|_{t=0}.$$

The *isotropy subgroup* of $z \in M$ is $G_z := \{g \in G | g \cdot z = z\}$. An action is *free* if all of the isotropy subgroups G_z are trivial.

An action is *proper* if the map $(g, z) \mapsto (z, g \cdot z)$ is proper (i.e. the preimage of every compact set is compact). Note that this is always the case if G is compact. A key elementary property of proper actions is that all isotropy subgroups are compact. If G acts properly and freely on M, then M/G has a unique smooth structure such that $\pi_G : M \to M/G$ is a submersion (in fact, π_G is a principal bundle). One useful consequence is that for every $z \in M$, we have $\ker T_z\pi_G = T_z(G \cdot z) = \mathfrak{g} \cdot z$.

Given a G action $\Phi : G \times Q \to Q$, the group G has a *tangent lift* action on TQ, given by $g \cdot v = T\Phi_g(v)$, and a *cotangent lift* action on T^*Q, given by $g \cdot \alpha = (T\Phi_{g^{-1}})^*\alpha$. In this context, the space Q called the *configuration space* or *base space*. The tangent or cotangent lift of a proper (resp. free) action is proper (resp. free). For any $q \in Q$, the isotropy group G_q is called the *configuration space isotropy* of any point $v \in T_qQ$ or $z \in T_q^*Q$. The cotangent bundle T^*Q has a canonical symplectic form, given in given local coordinates by $\omega = \mathrm{d}q^i \wedge \mathrm{d}p_i$. Every cotangent-lifted action on T^*Q is symplectic with respect to this symplectic form and has an Ad^*-equivariant momentum map given by $\langle J(\alpha_q), \xi\rangle = \langle \alpha_q, \xi \cdot q\rangle$.

1.2. Palais' slice theorem

Let K be a Lie subgroup of G, and S is a manifold on which K acts. Consider the following two left actions on $G \times S$:

$$\begin{aligned} &K \text{ acts by } \textit{twisting}: \quad k \cdot (g, s) = (gk^{-1}, k \cdot s)\\ &G \text{ acts by left } \textit{multiplication}: \quad \gamma \cdot (g, s) = (\gamma g, s). \end{aligned} \tag{1}$$

It is easy to show that these actions are free and proper and commute. The *twisted product* $G \times_K S$ is the quotient of $G \times S$ by the twist action of K. It is a smooth manifold; in fact $G \times_K S \to G/K$ is the vector bundle associated to the K action on S. The left multiplication action of G commutes with the twist action and drops to a smooth G action on $G \times_K S$, given by $\gamma \cdot [g, s]_K = [\gamma g, s]_K$.

Now consider a G action on a manifold M, and a point $z \in M$, and let $K = G_z$ be the isotropy subgroup of z. A *tube* for the G action at z is a G-equivariant diffeomorphism from some twisted product $G \times_K S$ to an open neighbourhood of $G \cdot z$ in M, that maps $[e, 0]_K$ to z. The space S may be embedded in $G \times_K S$ as $\{[e, s]_K : s \in S\}$; the image of the latter by the tube is called a *slice*.

The slice theorem of Palais [28] states that tubes always exist for smooth proper actions of a Lie group G on manifold M. One version of the theorem is as follows. Given $z \in M$, with isotropy group $K = G_z$, there always exists a G-invariant Riemannian metric on a neighbourhood of z. Let N be the orthogonal complement $\mathfrak{g} \cdot z$. Then there exists a

K-invariant neighbourhood S of 0 in N such that the map

$$\begin{aligned} \tau : G \times_K S &\longrightarrow M \\ [g, s]_K &\longmapsto g \cdot \exp_z s \end{aligned}$$

(where $\exp_z$ is the Riemannian exponential) is a tube for the G action at z. The K-invariant complement N to $\mathfrak{g} \cdot z$ is sometimes called a *linear slice* to the G action at z. The twisted product $G \times_K N$ may be identified with the normal bundle to the orbit $G \cdot z$. If the G action is linear, then we can replace "$\exp_z s$" with "$(z + s)$" in the above statement, and S may be chosen to be any neighbourhood of 0 such that τ is injective.

1.3. Configuration space slices

Consider a Lagrangian $L : TQ \to \mathbb{R}$, invariant under a proper tangent-lifted action of a Lie group G. Let $q_0 \in Q$ and $K = G_{q_0}$.

We can apply Palais's slice theorem around q_0, giving a tube

$$\begin{aligned} \tau : G \times_K S &\longrightarrow Q \\ [g, s]_K &\longmapsto g \cdot \exp_{q_0} s \end{aligned} \tag{2}$$

If Q is an open subset of a vector space, with G acting linearly, as, for example, in gravitational and molecular N-body problems, then S can be identified with a neighbourhood of the origin in a linear subspace of Q itself, and the tube defined by $\tau([g, s]_K) = g \cdot (q_0 + s)$. In any case, pulling back τ by the projection $\pi_K : G \times S \to G \times_K S$ gives a map $\tau \circ \pi_K : G \times S \to Q$ which we regard as degenerate "parametrisation" of Q in a neighbourhood of q, defining the "slice coordinates" (g, s). This parametrisation is semi-global in the sense that it is global in the group direction and local in the slice direction. The tangent and cotangent lifts of $\pi_K \circ \tau$ give parametrisations $T(G \times S) \to TQ$ and $T^*(G \times S) \to T^*Q$. In this paper we will describe mechanical systems on TQ and T^*Q, with configurations in the neighbourhood of the group orbit $G \cdot q_0$, by pulling them back to $T(G \times S)$ and $T^*(G \times S)$. We now describe the actions of G and K on these spaces.

Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K. Throughout the paper we identify TG with $G \times \mathfrak{g}$ and T^*G with $G \times \mathfrak{g}^*$ using the trivialisations given by:

$$\begin{aligned} TG &\xrightarrow{\cong} G \times \mathfrak{g} \quad \text{and} \quad & T^*G &\xrightarrow{\cong} G \times \mathfrak{g}^* \\ TL_g\zeta &\longmapsto (g, \zeta) & T^*L_{g^{-1}}\mu &\longmapsto (g, \mu) \end{aligned}$$

where L_g is left multiplication by g. Similarly, we make the identifications

$$T(G \times S) \cong TG \times TS \cong G \times \mathfrak{g} \times TS$$

$$T^*(G \times S) \cong T^*G \times T^*S \cong G \times \mathfrak{g}^* \times T^*S$$

Note that TS and T^*S are trivial, as S is a subset of a vector space. We write elements of TS as $(s, \dot{s})$ and elements of T^*S as (s, σ). The left multiplication action of G and twist action of K on $G \times S$ lift to free, proper, commuting actions on $T(G \times S)$ and $T^*(G \times S)$. In the above trivialisations, the lifted actions are:

$$\text{tangent lifted twist} : k \cdot (g, \zeta, s, \dot{s}) = (gk^{-1}, \mathrm{Ad}_k \zeta, k \cdot s, k \cdot \dot{s})$$

$$\text{cotangent lifted twist} : k \cdot (g, \mu, s, \sigma) = (gk^{-1}, \mathrm{Ad}^*_{k^{-1}} \mu, k \cdot s, k \cdot \sigma)$$

$$\text{tangent lifted left multiplication} : \gamma \cdot (g, \zeta, s, \dot{s}) = (\gamma g, \zeta, s, \dot{s})$$

$$\text{cotangent lifted left multiplication} : \gamma \cdot (g, \mu, s, \sigma) = (\gamma g, \mu, s, \sigma).$$

The corresponding infinitesimal actions of $\xi \in \mathfrak{k}$ and $\eta \in \mathfrak{g}$ are:

$$\text{tangent lifted twist} : \xi \cdot (g, \zeta, s, \dot{s}) = (-\xi, \mathrm{ad}_\xi \zeta, \xi \cdot s, \xi \cdot \dot{s})$$

$$\text{cotangent lifted twist} : \xi \cdot (g, \mu, s, \sigma) = (-\xi, -\mathrm{Ad}^*_\xi \mu, \xi \cdot s, \xi \cdot \sigma)$$

$$\text{tangent lifted left multiplication} : \eta \cdot (g, \zeta, s, \dot{s}) = (\mathrm{Ad}_{g^{-1}} \eta, 0, 0, 0)$$

$$\text{cotangent lifted left multiplication} : \eta \cdot (g, \mu, s, \sigma) = (\mathrm{Ad}_{g^{-1}} \eta, 0, 0, 0).$$

The cotangent-lifted actions have the following momentum maps, with respect to the canonical symplectic form on $T^*(G \times S)$:

$$\text{twist} : J_K(g, \mu, s, \sigma) = -\mu|_{\mathfrak{k}} + J_S(s, \sigma) = -\mu|_{\mathfrak{k}} + s \diamondsuit \sigma \in \mathfrak{k}^*$$

$$\text{left multiplication} : J_G(g, \mu, s, \sigma) = \mathrm{Ad}^*_{g^{-1}} \mu \in \mathfrak{g}^*,$$

where $s \diamondsuit \sigma$ is defined by $\langle s \diamondsuit \sigma, \xi \rangle = \langle \sigma, \xi \cdot s \rangle$ for all $\xi \in \mathfrak{k}$ and is the momentum map $J_S : T^*S \to \mathfrak{k}^*$ for the action of K on T^*S. The momentum map J_K is equivariant with respect to the twist action of K and invariant under the left multiplication action of G, while J_G is equivariant with respect to the action of G and invariant under the twist action.

For simplicity of notation, we will sometimes identify Q with $G \times_K S$ and q_0 with $[e, 0]_K$.

2. The Lagrangian side

The goal of this section is to describe the tangent bundle $T(G \times_K S)$ and any Lagrangian system on it, using a parametrisation by $G \times \mathfrak{k}^\perp \times TS \subset T(G \times S)$.

Fix a K-invariant complement of $\mathfrak{k}$ in $\mathfrak{g}$, which we denote $\mathfrak{k}^\perp$ (such a complement can always be found by averaging over K, since K is compact). Consider the projection

$\pi_K : G \times S \to G \times_K S$. Its tangent map is a K-invariant G-equivariant surjection. If we describe points in $T(G \times_K S)$ as $T\pi_K(g, \zeta, s, \dot{s})$, then we have two kinds of degeneracy in our coordinates: first, (g, s) is not uniquely determined by $\pi_K(g, s)$; second, given a choice of (g, s), the tangent vector $(\zeta, \dot{s}) \in T_{(g,s)}(G \times S)$ is not uniquely determined because $T_{(g,s)}\pi_K$ has a kernel,

$$\ker T_{(g,s)}\pi_K = \mathfrak{k} \cdot (g, s) = \{(-\xi, \xi \cdot s) | \xi \in \mathfrak{k}\}.$$

Note that $\mathfrak{k}^\perp \times S$ is complementary to $\ker T_{(g,s)}\pi_K$ for every (g, s). Therefore we can eliminate the second kind of degeneracy in our coordinates by restricting $T\pi_K$ to $G \times \mathfrak{k}^\perp \times TS \subset T(G \times S)$. Our new parametrisation of $T(G \times_K S)$ is

$$T\pi_K : G \times \mathfrak{k}^\perp \times TS \longrightarrow T(G \times_K S).$$

Note that for any $(g, s) \in G \times S$, the map $T_{(g,s)}\pi_K$ is an isomorphism from $\mathfrak{k}^\perp \times T_s S$ to $T_{[g,s]_K}(G \times_K S)$. Composing π_K with the tube τ from Eq. (2) gives a map $(g, s) \mapsto g \cdot \exp_{q_0} s$. Differentiating this gives

$$\begin{aligned} T(\tau \circ \pi_K) &: G \times \mathfrak{k}^\perp \times TS \longrightarrow TQ \\ (g, \zeta, s, \dot{s}) &\longmapsto g \cdot (\zeta \cdot \exp_{q_0} s + T_s \exp_{q_0}(\dot{s})) \end{aligned} \tag{3}$$

This formalises the observation that $T_q Q \cong \mathfrak{g} \cdot q \oplus S$ near the point at which the slice S is defined.

Since $G \times \mathfrak{k}^\perp \times TS$ and $T\pi_K$ are K-invariant, the map $T\pi_K$ descends to the quotient by K,

$$\overline{T\pi_K} : (G \times \mathfrak{k}^\perp \times TS)/K \longrightarrow T(G \times_K S).$$

It is easily checked that this map is a G-equivariant diffeomorphism.

The K action on $G \times \mathfrak{k}^\perp \times TS$ is exactly the twist action on $G \times (\mathfrak{k}^\perp \times TS)$ given by the adjoint action on K on $\mathfrak{k}^\perp$ and the tangent-lifted action of K on TS. Thus $\overline{T\pi_K}$ may be written

$$\overline{T\pi_K} : G \times_K (\mathfrak{k}^\perp \times TS) \longrightarrow T(G \times_K S)$$

$$[e, 0, 0]_K \longmapsto 0 \in [e, 0]_K$$

and we see that $\overline{T\pi_K}$ is actually a tube for $T(G \times_K S)$ around $0 \in [e, 0]_K$.

Now let $L : T(G \times_K S) \to \mathbb{R}$ be a smooth Lagrangian. We define $\tilde{L} : T(G \times S) \to \mathbb{R}$ by $\tilde{L} = L \circ T\pi_K$ and $\overline{L} : G \times \mathfrak{k}^\perp \times TS \to \mathbb{R}$ as the restriction of $\tilde{L}$ to $G \times \mathfrak{k}^\perp \times TS$. Using Hamilton's principle one can prove:

Proposition 1. *If L is a regular Lagrangian on $T(G \times_K S)$ then the Euler–Lagrange equations for $\overline{L}$ have solutions. Furthermore, a curve $\overline{c} : [a, b] \to G \times \mathfrak{k}^\perp \times TS$ is a solution for the Euler–Lagrange equations for $\overline{L}$ if and only if $\overline{c}$ projects to a curve on $T(G \times_K S)$ which is a solution for the Euler–Lagrange equations for L.*

3. The Hamiltonian side

In this section we describe a parametrisation of $T^*(G \times_K S)$ by $G \times \mathfrak{k}^\circ \times T^*S$ that is dual to the parametrisation of $T(G \times_K S)$ by $G \times \mathfrak{k}^\perp \times TS$ described in the previous section. Here $\mathfrak{k}^\circ$ is the annihilator of $\mathfrak{k}$ in $\mathfrak{g}$. Note that our choice of splitting $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{k}^\perp$ induces a dual splitting $\mathfrak{g} = \left(\mathfrak{k}^\perp\right)^\circ \oplus \mathfrak{k}^\circ \cong \mathfrak{k}^* \oplus (\mathfrak{k}^\perp)^*$.

Recall that J_K is the momentum map for the cotangent-lifted twist action of K on $T^*(G \times S)$. By regular cotangent bundle reduction, the symplectic reduced space at zero $J_K^{-1}(0)/K$ is symplectomorphic to $T^*((G \times S)/K) = T^*(G \times_K S)$; this is a special case of results due to Satzer and Marsden (see [1]). The isomorphism $\overline{\rho} : J_K^{-1}(0)/K \to T^*(G \times_K S)$ is the quotient by K of the map

$$\rho : J_K^{-1}(0) \to T^*(G \times_K S), \quad \langle \rho(p), T\pi_K(v)\rangle = \langle p, v\rangle. \tag{4}$$

(This map ρ is a sort of "push-forward" by π_K, though π_K is not injective.) Let $H : T^*(G \times_K S) \to \mathbb{R}$ be a Hamiltonian, and define $\tilde{H} : J_K^{-1}(0) \to \mathbb{R}$ by $\tilde{H} = H \circ \rho$. The cotangent bundle reduction theorem implies that, given any K-invariant extension H^{ext} of $\tilde{H}$, the Hamiltonian vector field $X_{H^{\text{ext}}}$ projects down to the original Hamiltonian vector field X_H, in the sense that $T\rho(X_{H^{\text{ext}}}(z)) = X_H(\rho(z))$ for every $z \in J_K^{-1}(0)$. Both $X_{H^{\text{ext}}}$ and X_H are defined with respect to the canonical symplectic forms on the relevant cotangent bundles. Finally, it is easy to check that ρ and $\overline{\rho}$ are G-equivariant.

The level set $J_K^{-1}(0)$ is conveniently parametrised as follows,

$$\varphi : G \times \mathfrak{k}^\circ \times T^*S \to J_K^{-1}(0) \subset G \times \mathfrak{g}^* \times T^*S$$

$$(g, \nu, s, \sigma) \mapsto (g, \nu + s \diamondsuit \sigma, s, \sigma).$$

The formula $J_K(g, \mu, s, \sigma) = -\mu|_{\mathfrak{k}} + s \diamondsuit \sigma$ shows that φ is well-defined and surjective. Its inverse is $\varphi(g, \mu, s, \sigma) = (g, \mu|_{\mathfrak{k}^\perp}, s, \sigma)$. It is clear that φ is G- and K- equivariant. We define $\overline{H} = \tilde{H} \circ \varphi = H \circ \rho \circ \varphi$.

Since φ is G- and K-equivariant, it descends to a G-equivariant map

$$\overline{\varphi} : (G \times \mathfrak{k}^\circ \times T^*S)/K \to J_K^{-1}(0)/K.$$

The K action on $G \times \mathfrak{k}^\perp \times T^*S$ is the same as the twist action on $G \times (\mathfrak{k}^\perp \times T^*S)$ given by the adjoint action on K on $\mathfrak{k}^\circ$ and the cotangent-lifted action of K on TS. Thus the domain of $\overline{\varphi}$ may be written as $G \times_K (\mathfrak{k}^\circ \times T^*S)$. In summary, writing Π_K for the projection $T^*(G \times S) \to T^*(G \times S)/K$, we have

$$\begin{array}{ccccc} G \times \mathfrak{k}^\circ \times T^*S & \xrightarrow{\varphi} & J_K^{-1}(0) & & \\ \Pi_K \downarrow & & \Pi_K \downarrow & \overset{\rho}{\searrow} & \\ G \times_K (\mathfrak{k}^\circ \times T^*S) & \xrightarrow{\bar{\varphi}} & J_K^{-1}(0)/K & \xrightarrow{\bar{\rho}} & T^*(G \times_K S) \end{array}$$

The composition $\overline{\rho} \circ \overline{\varphi}$ is a special case of the symplectic tube given by the cotangent bundle slice theorem [32], which is a constructive version of the Hamiltonian slice theorem, also known as the Marle–Guillemin–Sternberg normal form (see [12,6]). The tube is based at the point $0 \in T^*_{q_0}(G \times_K S)$, where $q_0 = [e, 0]_k = \pi_K(e, 0)$, meaning that $\overline{\rho} \circ \overline{\varphi}([e, 0, 0, 0]_K) = q_0$.

4. Legendre transforms

In the previous two sections we have found twisted parametrisations of TQ and T^*Q respecting a given symmetry. Given an original symmetric Lagrangian on TQ or symmetric Hamiltonian on T^*Q, we have defined "lifted Lagrangians" and "lifted Hamiltonians" using the new parametrisations. In the present section we consider the situation when the original Hamitonian is obtained from the original Lagrangian as the Legendre transform of the energy. We obtain Legendre transforms of the lifted Lagrangians, and show that lifted Hamiltonians are related to the lifted Lagrangians in the natural way.

Recall that, for any smooth function $L : TQ \to \mathbb{R}$, the Legendre transform of L is the map $\mathbb{F}L : TQ \to T^*Q$ defined by

$$\mathbb{F}L(q, v) = \frac{\partial L}{\partial v}(q, v) \quad \forall q \in Q, v \in T_q Q$$

The Lagrangian L is said to be *regular* if the derivative of $\mathbb{F}L$ has maximal rank everywhere, and to be *hyperregular* if $\mathbb{F}L$ is a diffeomorphism between TQ and T^*Q. The *energy* function associated to L is $E : TQ \to \mathbb{R}$ given by $E(v) = \mathbb{F}L(v) \cdot v - L(v)$. If L is G-invariant then $\mathbb{F}L$ is G-equivariant and E is G-invariant. In classical theory, given a hyperregular Lagrangian, the Hamiltonian H of a mechanical system is obtained as $H := E \circ \mathbb{F}L^{-1}$.

Let $L : T(G \times_K S) \to \mathbb{R}$ be a G-invariant Lagrangian. In Section 2 we defined two related "lifted" Lagrangians: $\tilde{L} : T(G \times S) \to \mathbb{R}$ given by $\tilde{L} = L \circ T\pi_K$; and $\overline{L}$ defined as the restriction of $\tilde{L}$ to $G \times \mathfrak{k}^\perp \times TS$. The Legendre transform of $F\tilde{L}$ has codomain $T^*(G \times S)$, though we will see shortly that it is not surjective. Identifying $\left(\mathfrak{k}^\perp\right)^*$ with $\mathfrak{k}^\circ$, the Legendre transform of $F\overline{L}$ has codomain $G \times \mathfrak{k}^\circ \times T^*S$; we will see that it *is* surjective. The following diagram summarises our main definitions so far; all of the upwards arrows are inclusions.

$$\begin{array}{ccccc}
T(G \times S) & \xrightarrow{\mathbb{F}\tilde{L}} & T^*(G \times S) & & \\
\uparrow & & \uparrow & \nwarrow & \\
G \times \mathfrak{k}^\perp \times TS & \xrightarrow{\mathbb{F}\bar{L}} & G \times \mathfrak{k}^\circ \times T^*S & \xrightarrow{\varphi} & J_K^{-1}(0) \\
T\pi_K \downarrow & & & \rho\swarrow & \\
T(G \times_K S)(\mathfrak{k}^\circ \times T^*S) & \xrightarrow{\mathbb{F}L} & T^*(G \times_K S) & &
\end{array} \tag{5}$$

Proposition 2. *Let L be a Lagrangian on $T(G \times_K S)$, and let $\tilde{L}$ and $\overline{L}$ be defined as above. Then*

(1) $\langle \mathbb{F}\tilde{L}(v), w\rangle = \langle \mathbb{F}L(T\pi_K(v)), T\pi_K(w)\rangle$ *for all $v, w \in T(G \times S)$.*
(2) *The image of $\mathbb{F}\tilde{L}$ is contained in $J_K^{-1}(0)$ and $\rho \circ \mathbb{F}\tilde{L} = \mathbb{F}L \circ T\pi_K$.*
(3) *For every $v \in T(G \times S)$, the corank of $T_v\mathbb{F}\tilde{L}$ is at least* $\dim K$, *with equality if and only if L is regular at $T\pi_K(v)$.*

(4) $\varphi \circ \mathbb{F}\bar{L}$ *equals the restriction of* $\mathbb{F}\tilde{L}$ to $G \times \mathfrak{k}^{\perp} \times TS$.
(5) *If L is hyperregular then* $\mathbb{F}\bar{L}$ *is a diffeomorphism.*

Proof.

(1) For every $(g, s) \in G \times S$ and every $v, w \in T_{(g,s)}(G \times S)$,

$$\langle \mathbb{F}\tilde{L}(v), w\rangle = \left\langle \frac{\partial (L \circ T\pi_K)}{\partial v}(v), w \right\rangle = \left\langle \frac{\partial L}{\partial v}(T\pi_K(v)) \circ TT\pi_K(v), w \right\rangle.$$

By linearity of $T\pi_K$ on fibres, this equals $\langle (\partial L/\partial v)(T\pi_K(v)), T\pi_K(w)\rangle$.

(2) By the first claim, $\langle \mathbb{F}\tilde{L}(v), \xi \cdot (g, s)\rangle = 0$ for all $v \in T_{(g,s)}(G \times S)$ and all $\xi \in \mathfrak{k}$, which implies $J_K(\mathbb{F}\tilde{L}(v)) = 0$. By definition of ρ,

$$\langle \rho \circ \mathbb{F}\tilde{L}(v), T\pi_K(w)\rangle = \langle \mathbb{F}\tilde{L}, w\rangle = \langle \mathbb{F}L(T\pi_K(v)), T\pi_K(w)\rangle,$$

for all $v, w \in T(G \times S)$, which shows that $\rho \circ \mathbb{F}\tilde{L} = \mathbb{F}L \circ T\pi_K$.

(3) Let $v \in T(G \times S)$. It follows directly from Claim 2 that the corank of $T_v\mathbb{F}\tilde{L}$ is at least the codimension of $J_K^{-1}(0)$, which equals the dimension of K. The corank *equals* dim K if and only if $T_v\mathbb{F}\tilde{L}$ is onto $T_{\mathbb{F}\tilde{L}(v)} J_K^{-1}(0)$, which is equivalent to $T_v(\rho \circ \mathbb{F}\tilde{L})$ being surjective. In light of Claim 2, this is equivalent to the surjectivity of $T_v(\mathbb{F}L \circ T\pi_K)$. Since π_K is a submersion, this is equivalent to $T_v(\mathbb{F}L)$ being surjective, i.e. L being regular at v.

(4) Since $\bar{L}$ is the restriction of $\tilde{L}$ to $G \times \mathfrak{k}^{\perp} \times TS$, it follows that $\langle \mathbb{F}\bar{L}(v), w\rangle = \langle \mathbb{F}\tilde{L}(v), w\rangle$ for any $v, w \in G \times \mathfrak{k} \times TS$. Thus $\mathbb{F}\bar{L}(v)$ is the restriction of $\mathbb{F}\tilde{L}(v)$ to $G \times \mathfrak{k}^{\perp} \times TS$. In coordinates, if $\mathbb{F}\tilde{L}(v) = (g, \mu, s, \sigma)$, then $\mathbb{F}\bar{L}(v) = (g, \mu|_{\mathfrak{k}^{\perp}}, s, \sigma) = \varphi^{-1}(\mathbb{F}\tilde{L}(v))$.

(5) If L is hyperregular, then Claims 2 and 3 imply that that $\mathbb{F}\tilde{L}$ is a surjective submersion onto $J_K^{-1}(0)$. Suppose $\mathbb{F}\tilde{L}(v_1) = \mathbb{F}\tilde{L}(v_2)$. From Claim 1 and the surjectivity of $T\pi_K$, it follows that $\mathbb{F}L(T\pi_K(v_1)) = \mathbb{F}L(T\pi_K(v_2))$. Hyperregularity of L then implies that $T\pi_K(v_1) = T\pi_K(v_2)$. Since v_1 and v_2 must be in the same tangent fibre $T_{(g,s)}(G \times S)$, this is equivalent to $v_2 - v_1 \in \ker T_{(g,s)}\pi_K = \mathfrak{k} \cdot (g, s)$. If v_1 and v_2 are both in $G \times \mathfrak{k}^{\perp} \times TS$, this implies that $v_1 = v_2$. Hence the restriction of $\mathbb{F}\tilde{L}$ to $G \times \mathfrak{k}^{\perp} \times TS$ is a bijective submersion, and hence a diffeomorphism, onto $J_K^{-1}(0)$. From Claim 4, it follows that $\mathbb{F}\bar{L}$ is a diffeomorphism $G \times \mathfrak{k}^{\circ} \times T^*S$. □

Proposition 3. *Let L be a hyperregular Lagrangian on* $T(G \times_K S)$, *with energy function E, and let* $H = E \circ (\mathbb{F}L)^{-1}$. *Let* $\tilde{L}$ *and* $\bar{L}$ *be defined as above, with corresponding energy functions* $\tilde{E}$ *and* $\bar{E}$. *Let* $\tilde{H}$ *and* $\bar{H}$ *be as defined in* Section 3, i.e. $\tilde{H} = H \circ \rho$ *and* $\bar{H} = \tilde{H} \circ \varphi$. *Then*:

(1) $\tilde{E} = E \circ T\pi_K$
(2) $\bar{E} = \tilde{E}|_{G \times \mathfrak{k}^{\perp} \times TS}$
(3) $\tilde{H} \circ \mathbb{F}\tilde{L} = \tilde{E}$
(4) $\bar{H} = \bar{E} \circ (\mathbb{F}\bar{L})^{-1}$

Proof.

(1) For every $v \in T(G \times S)$, using Claim 1 of Proposition 2, we have

$$\begin{aligned}\tilde{E}(v) &= \langle \mathbb{F}\tilde{L}(v), v\rangle - \tilde{L}(v) = \langle \mathbb{F}L(T\pi_K(v)), (T\pi_K(v))\rangle - L(T\pi_K(v)) \\ &= E(T\pi_K(v)).\end{aligned}$$

(2) For every $v \in G \times \mathfrak{k}^\perp \times TS$, we have $\bar{E}(v) = \langle \mathbb{F}\bar{L}(v), v\rangle - \bar{L}(v)$. Using the definition of $\bar{L}$ as the restriction of $\tilde{L}$ and Claim 3 of Proposition 2, it follows that $\bar{E}(v) = \langle \varphi^{-1} \circ \mathbb{F}\tilde{L}(v), v\rangle - \tilde{L}(v)$. Now $\varphi^{-1}(g, \mu, s, \sigma) = (g, \mu|_{\mathfrak{k}^\perp}, s, \sigma)$, and $v \in G \times \mathfrak{k}^\perp \times TS$, so $\langle \varphi^{-1} \circ \mathbb{F}\tilde{L}(v), v\rangle = \langle \mathbb{F}\tilde{L}(v), v\rangle$. It follows that $\bar{E}(v) = \tilde{E}(v)$.

(3) From Claim 2 of Proposition 2, $(\mathbb{F}L)^{-1} \circ \rho \circ \mathbb{F}\tilde{L} = T\pi_K$. Hence

$$\tilde{H} \circ \mathbb{F}\tilde{L} = H \circ \rho \circ \mathbb{F}\tilde{L} = E \circ (\mathbb{F}L)^{-1} \circ \rho \circ \mathbb{F}\tilde{L} = E \circ T\pi_K = \tilde{E}$$

(by Claim 1).

(4) Using Claims 2 and 3 of the present proposition, we have $\bar{E} \circ (\mathbb{F}\bar{L})^{-1} = \tilde{E} \circ (\mathbb{F}\bar{L})^{-1} = \tilde{H} \circ \mathbb{F}\tilde{L} \circ (\mathbb{F}\bar{L})^{-1}$. By Claim 3 of Proposition 2, this equals $\tilde{H} \circ \varphi$, which equals $\bar{H}$ by definition. $\square$

We have thus found "the link" between our lifted Lagrangians and Hamiltonians. We will next study the lifted Hamiltonian system, with emphasis on relative equilibria, and then use our lifted Legendre transform $\mathbb{F}\bar{L}$ to study the relative equilibria of simple mechanical systems.

5. Hamilton's equations and relative equilibria

In this section we outline a calculation of Hamilton's equations in the twisted parametrisation of T^*Q given in Section 3. The result will be a special case of the *bundle equations* or *reconstruction equations* (see Remark 4). We then apply these to give conditions for the existence of relative equilibria.

Recall that, given any K-invariant extension H^{ext} of the function $\tilde{H} := H \circ \rho$, the restriction of the Hamiltonian vector field $X_{H^{\text{ext}}}$ to $J_K^{-1}(0)$ projects down to the orignal Hamiltonian vector field X_H on T^*Q. We now choose a particular extension H^{ext}. Recall the diffeomorphism $\varphi : G \times \mathfrak{k}^\circ \times T^*S \to J_K^{-1}(0)$, $\varphi(g, \mu_\perp, s, \sigma) = (g, \mu_\perp + s \diamondsuit \sigma, s, \sigma)$ (Proposition 4.3). Its inverse map is simply $\varphi^{-1}(g, \mu, s, \sigma) = (g, \mu|_{\mathfrak{k}^\perp}, s, \sigma)$. Define H^{ext} to be the pull-back of $\bar{H}$ by the projection $P_K^* : G \times \mathfrak{g} \times S \times S^* \to G \times \mathfrak{k}^\circ \times S \times S^*$ given by $P_K^*(g, \mu, s, \sigma) = (g, \mu|_{\mathfrak{k}^\perp}, s, \sigma)$; equivalently

$$H^{\text{ext}}(g, \mu, s, \sigma) = \bar{H}(g, \mu|_{\mathfrak{k}^\perp}, s, \sigma)\,.$$

Since P_K^* and φ are G- and K-invariant, so is H^{ext}. It is well-known that Hamilton's equations on $T^*(G \times S)$, with respect to the canonical symplectic form and the left trivialisation

$T^*(G \times S) \cong G \times \mathfrak{g}^* \times S \times S^*$, take the following form (see for example [14]),

$$\begin{cases} \zeta = \dfrac{\partial H^{\text{ext}}}{\partial \mu} \\ \dot{\mu} = \text{ad}^*_{\zeta}\mu \\ \dot{s} = \dfrac{\partial H^{\text{ext}}}{\partial \sigma} \\ \dot{\sigma} = -\dfrac{\partial H^{\text{ext}}}{\partial s} \end{cases} \tag{6}$$

It is clear from the definition of H^{ext} that H^{ext} may be replaced by $\bar{H}$ in the third and fourth equations. Splitting ζ into $\mathfrak{k}$ and $\mathfrak{k}^{\perp}$ components and μ into $\mathfrak{k}^*$ and $\mathfrak{k}^{\circ}$ components, we find that the first equation above splits into $\zeta_{\mathfrak{k}} = \partial H^{\text{ext}}/\partial \mu_{\mathfrak{k}} = 0$ and $\zeta_{\perp} = \partial H^{\text{ext}}/\partial \mu_{\perp} = \partial \bar{H}/\partial \mu_{\perp}$.

Since H^{ext} is K-invariant, Noether's theorem implies that $J_K^{-1}(0)$ is an invariant manifold for the flow of $X_{H^{\text{ext}}}$. It is thus valid to compute the pull back $\varphi^* X_{H^{\text{ext}}}$ on $G \times \mathfrak{k}^{\circ} \times T^*S$ by applying the change of variable $\mu = \mu_{\perp} + s \diamondsuit \sigma$. We introduce the notation $\overline{\text{ad}}^*_{\zeta}\mu := (\text{ad}^*_{\zeta}\mu)|_{\mathfrak{k}^{\perp}}$, incorporating a projection onto $\mathfrak{k}^{\circ}$. The $\mu_{\perp}$ equation of the pull-backed vector field is

$$\dot{\mu}_{\perp} = \overline{\text{ad}}^*_{\partial \bar{H}/\partial \mu_{\perp}}(\mu_{\perp} + s \diamondsuit \sigma).$$

Observe that since $\partial \bar{H}/\partial \mu_{\perp} \in \mathfrak{k}^{\perp}$ and $s \diamondsuit \sigma \in \mathfrak{k}^*$ we have, for every $\xi \in \mathfrak{k}$,

$$\langle \text{ad}^*_{\partial \bar{H}/\partial \mu_{\perp}}(s \diamondsuit \sigma), \xi \rangle = -\left\langle \text{ad}^*_{\xi}(s \diamondsuit \sigma), \frac{\partial \bar{H}}{\partial \mu_{\perp}} \right\rangle = 0.$$

Thus $\overline{\text{ad}}^*_{\partial \bar{H}/\partial \mu_{\perp}}(s \diamondsuit \sigma) = \text{ad}^*_{\partial \bar{H}/\partial \mu_{\perp}}(s \diamondsuit \sigma)$. Hence the vector field $\varphi^* X_{H^{\text{ext}}}$ on $G \times \mathfrak{k}^{\circ} \times T^*S$ is:

$$\begin{cases} \zeta_{\mathfrak{k}} = 0 \\ \zeta_{\perp} = \dfrac{\partial \bar{H}}{\partial \mu_{\perp}} \\ \dot{\mu}_{\perp} = \overline{\text{ad}}^*_{\partial \bar{H}/\partial \mu_{\perp}}\mu_{\perp} + \text{ad}^*_{\partial \bar{H}/\partial \mu_{\perp}} s \diamondsuit \sigma \\ \dot{s} = \dfrac{\partial \bar{H}}{\partial \sigma} \\ \dot{\sigma} = -\dfrac{\partial \bar{H}}{\partial s} \end{cases} \tag{7}$$

Remark 4. The above equations are actually a special case of the *bundle equations* (*reconstruction equations*) for Hamiltonian systems [20,25,31], based at the phase space point $0 \in T^*_{q_0}Q$. We have derived them directly here rather than apply the general theory.

Remark 5. The above equations do *not* describe the restriction of the vector field $X_{H^{\text{ext}}}$ to $G \times \mathfrak{k}^\circ \times S \times S^*$. Instead, they describe the restriction of $X_{H^{\text{ext}}}$ to $J_K^{-1}(0)$, described in the coordinates $(g, \mu_\perp, s, \sigma)$ defined by φ.

Recall the surjective submersion $\rho : J_K^{-1}(0) \to T^*(G \times_K S)$ from cotangent bundle reduction (see Eq. (4)). Recall that a point $\rho(z) \in T^*(G \times_K S)$ is a relative equilibrium if and only if $X_H(\rho(z)) = \eta \cdot \rho(z)$ for some $\eta \in \mathfrak{g}$. We say that such a relative equilibrium has *velocity* η, even though η is not well-defined in the presence of isotropy. Since $\overline{\rho} : J_K^{-1}(0)/K \to T^*(G \times_K S)$ is a diffeomorphism, it follows that $\ker T_z\rho = \mathfrak{k} \cdot z$. So $\rho(z)$ is a relative equilibrium with velocity η if and only if

$$X_{H^{\text{ext}}}(z) = \eta^G \cdot z + \xi^K \cdot z \tag{8}$$

for some $\xi \in \mathfrak{k}$, where we have used superscripts to distinguish the G and K actions. Since φ is G- and K-equivariant, we can pull back this condition to $G \times \mathfrak{k}^\circ \times T^*S$ as follows: $\rho(\varphi(z))$ is a relative equilibrium if and only if $\varphi^* X_{H^{\text{ext}}}(z) = \eta^G \cdot z + \xi^K \cdot z$. Using the formulae in Eq. (7) for the vector field $X_{H^{\text{ext}}}$ and using Lemma 2.1 for the expressions of the infinitesimal actions $\eta^G \cdot z$ and $\xi^K \cdot z$, we obtain

$$\begin{cases} \mathbb{P}_{\mathfrak{k}}(\mathrm{Ad}_{g^{-1}}\eta) - \xi = 0 \\ \mathbb{P}_{\mathfrak{k}^\perp}(\mathrm{Ad}_{g^{-1}}\eta) = \dfrac{\partial \bar{H}}{\partial \mu_\perp} \\ -\mathrm{ad}^*_\xi \mu_\perp = \overline{\mathrm{ad}}^*_{\partial \bar{H}/\partial \mu_\perp}\mu_\perp + \mathrm{ad}^*_{\partial \bar{H}/\partial \mu_\perp} s \diamondsuit \sigma \\ \xi \cdot s = \dfrac{\partial \bar{H}}{\partial \sigma} \\ \xi \cdot \sigma = -\dfrac{\partial \bar{H}}{\partial s} \end{cases} \tag{9}$$

Remark 6. The last three equation of (9) are the relative equilibria conditions on the reduced space $J_K^{-1}(0)/G$, whereas the first two equations lift the dynamics back to the unreduced space $J_K^{-1}(0) \subset G \times \mathfrak{g}^* \times T^*S$. Note that the reduced space is a mixed Poisson-cotangent bundle space with coupled dynamics. The reduced space is subject to a residual symmetry given by the isotropy group K action on the slice S. In particular this implies that in the reduced space the relative equilibria are not points, but dynamical orbits, relative equilibria themselves with respect to the K action on the slice.

6. Simple mechanical systems

In this section we consider the special case of a *simple mechanical system*, which is one in which the Lagrangian $L : TQ \to \mathbb{R}$ has the form

$$L(q, v_q) = \tfrac{1}{2}\mathbb{K}(v_q, v_q) - V(q) \tag{10}$$

for some G-invariant Riemannian metric $\mathbb{K}$ on TQ, called the *kinetic energy*, and some G-invariant potential $V : Q \to \mathbb{R}$. We compute the Lagrangian $\bar{L}$ and Hamiltonian $\bar{H}$ for such systems and use $\overline{H}$ together with the results of the previous section to give conditions for the existence of relative equilibria of simple mechanical systems.

Let $q_0 \in Q$ have isotropy group $K = G_{q_0}$. Let $N = (\mathfrak{g} \cdot q_0)^{\perp}$ be the orthogonal complement to the tangent to the group orbit through q_0, with respect to the given metric. By applying Palais' slice theorem as in Section 1, we obtain, for some neighborhood S of 0 in N, a G-equivariant diffeomorphism $\tau : G \times_K S \to Q$. Note that $T_s S \cong N$ for any $s \in S$. Recall from Eq. (3) the parametrisation

$$\begin{aligned} &T(\tau \circ \pi_K) : G \times \mathfrak{k}^{\perp} \times TS \longrightarrow TQ \\ &(g, \zeta, s, \dot{s}) \longmapsto g \cdot (\zeta \cdot \exp_{q_0} s + T_s \exp_{q_0} \dot{s}) \end{aligned} \tag{11}$$

and recall that for any $(g, s) \in G \times S$, the map $T_{(g,s)}(\tau \circ \pi_K)$ is an isomorphism from $\mathfrak{k}^{\perp} \times N$ to $T_{\tau[g,s]_K} Q$. We will write the metric tensor in these coordinates. Since the metric is G-invariant, $\mathbb{K}(\tau[g, s]_K)$ depends only on s. For any s, we see that $\mathbb{K}(s)$ is a symmetric bilinear form on $\mathfrak{k}^{\perp} \times N$, which we represent as a matrix. This matrix can be written in block form, with respect to the splitting $T_{\tau[g,s]_K} Q \cong \mathfrak{k}^{\perp} \oplus N$, as follows:

$$\mathbb{K}(s) = \begin{pmatrix} \mathbb{I}_{\mathrm{r}}(s) & \mathbb{C}(s) \\ \mathbb{C}^T(s) & m(s) \end{pmatrix}$$

The block $\mathbb{I}_{\mathrm{r}}$ is called the *reduced locked inertia tensor*. It is related to the usual locked inertia tensor $\mathbb{I}$ by

$$\mathbb{I}(g \cdot \exp_{q_0} s)(\xi, \eta) = \mathbb{I}_{\mathrm{r}}(s)(\mathrm{Ad}_{g^{-1}}\xi, \mathrm{Ad}_{g^{-1}}\eta)$$

for any $\xi, \eta \in \mathfrak{k}^{\perp}$. The block $m(s)$ is called the *reduced mass*. The terminology comes from the fact that the kinetic energy matrix is often the mass matrix. Note that $\mathbb{I}_{\mathrm{r}}(s)$ and $m(s)$ are invertible. The block $\mathbb{C}(s)$ is called the *Coriolis tensor*. It couples the system, and is related to the usual Coriolis forces. Our choice of coordinates enforces $\mathbb{C}(0) = 0$, since $T_{(e,0)}(\tau \circ \pi_K)$ maps $\mathfrak{k}^{\perp} \times \{0\}$ to $\mathfrak{k}^{\perp} \cdot q_0 = \mathfrak{k} \cdot q_0$ and $\{0\} \times N$ to $T_0 \exp_{q_0}(N) = N = (\mathfrak{k} \cdot q_0)^{\perp}$. This will mean that the mechanical system in slice coordinates is decoupled at q_0.

The potential V can be written in slice coordinates as $V(s)$, since it is G-invariant. Thus the Lagrangian $\overline{L} : G \times \mathfrak{k}^{\perp} \times TS \to \mathbb{R}$ defined in Section 2 takes the form

$$\overline{L}(g, \zeta_{\perp}, s, \dot{s}) = L(T(\tau \circ \pi_K)(g, \zeta_{\perp}, s, \dot{s})) = \frac{1}{2}(\zeta_{\perp}\ \dot{s})\mathbb{K}(s) \begin{pmatrix} \zeta_{\perp} \\ \dot{s} \end{pmatrix} - V(s).$$

It follows that

$$\mathbb{F}\overline{L}_{(g,s)}(\zeta_{\perp}, \dot{s}) = \mathbb{K}(s) \begin{pmatrix} \zeta_{\perp} \\ \dot{s} \end{pmatrix} = \begin{pmatrix} \mathbb{I}_{\mathrm{r}}(s) & \mathbb{C}(s) \\ \mathbb{C}^T(s) & m(s) \end{pmatrix} \begin{pmatrix} \zeta_{\perp} \\ \dot{s} \end{pmatrix}. \tag{12}$$

The inverse of this Legendre transform can be written as follows, using the new matrices $\mathbb{A} := \mathbb{I}_{\rm r}^{-1}\mathbb{C}$ and $\mathbb{M} := m - \mathbb{C}^T\mathbb{I}_{\rm r}^{-1}\mathbb{C}$,

$$(\mathbb{F}\bar{L}_{(g,s)})^{-1}(\mu_\perp, \sigma) = \mathbb{K}(s)^{-1}\begin{pmatrix}\mu_\perp\\ \sigma\end{pmatrix} = \begin{pmatrix}\mathbb{I}_{\rm r}^{-1} + \mathbb{A}\mathbb{M}^{-1}\mathbb{A}^T & -\mathbb{A}\mathbb{M}^{-1}\\ -\mathbb{M}^{-1}\mathbb{A}^T & \mathbb{M}^{-1}\end{pmatrix}\begin{pmatrix}\mu_\perp\\ \sigma\end{pmatrix} \tag{13}$$

where $\mathbb{I}_{\rm r}^{-1}$, $\mathbb{A}$, and $\mathbb{M}$ are all functions of s.

We now compute the Hamiltonian $\bar{H} : G \times (\mathfrak{k}^\perp)^* \times T^*S \to \mathbb{R}$. Recall the formula $\bar{H} = \bar{E} \circ \left(\mathbb{F}\bar{L}\right)^{-1}$ from Proposition 3. For a simple mechanical L, the energy function is $\bar{E}(v) = \langle \mathbb{F}\bar{L}(v), v\rangle - L(v) = (1/2)v^T\mathbb{K}v + V$, so, just as in the free case, we have

$$\bar{H}(p) = \tfrac{1}{2}p^T\mathbb{K}^{-1}p + V.$$

Using Eq. (13), we find

$$\bar{H} = \tfrac{1}{2}\mu_\perp^T\mathbb{I}_{\rm r}^{-1}\mu_\perp + \tfrac{1}{2}(\sigma - \mathbb{A}^T\mu_\perp)^T\mathbb{M}^{-1}(\sigma - \mathbb{A}^T\mu_\perp) + V. \tag{14}$$

Note that

$$\left(\frac{\partial\bar{H}}{\partial\mu_\perp}, \frac{\partial\bar{H}}{\partial\sigma}\right) = \mathbb{K}(s)^{-1}\begin{pmatrix}\mu_\perp\\ \sigma\end{pmatrix} = (\mathbb{F}\bar{L}_{(g,s)})^{-1}(\mu_\perp, \sigma)$$

We will now compute the relative equilibrium conditions from Eq. (9) in the case of simple mechanical systems. The partial derivatives of $\bar{H}$ are as follows:

$$\begin{aligned}
\frac{\partial\bar{H}}{\partial\mu_\perp} &= \mathbb{I}_{\rm r}^{-1}\mu_\perp - \mathbb{A}\mathbb{M}^{-1}(\sigma - \mathbb{A}^T\mu_\perp)\\
\frac{\partial\bar{H}}{\partial\sigma} &= \mathbb{M}^{-1}(\sigma - \mathbb{A}^T\mu_\perp)\\
\frac{\partial\bar{H}}{\partial s} &= \frac{\mathrm{d}}{\mathrm{d}s}\left(\tfrac{1}{2}\mu_\perp^T\mathbb{I}_{\rm r}^{-1}\mu_\perp\right) - \mu_\perp^T\frac{\mathrm{d}\mathbb{A}}{\mathrm{d}s}\mathbb{M}^{-1}(\sigma - \mathbb{A}^T\mu)\\
&\quad + \frac{1}{2}(\sigma - \mathbb{A}^T\mu_\perp)^T\frac{\mathrm{d}\mathbb{M}^{-1}}{\mathrm{d}s}(\sigma - \mathbb{A}^T\mu_\perp) + \frac{\mathrm{d}V}{\mathrm{d}s}
\end{aligned}$$

Note that the last of the relative equilibrium conditions in Eq. (9) is $\xi \cdot \sigma = -\partial\bar{H}/\partial s$. We now show that this equation can be expressed in terms of the following function, which is a generalisation of both the amended and augmended potentials familiar from the free case.

Definition 7. The augmented-amended potential is the function

$$V_{\mu_\perp,\xi}(s) = \tfrac{1}{2}[\mu_\perp - \mathbb{C}(\xi\cdot s)]^T\mathbb{I}_{\rm r}^{-1}[\mu_\perp - \mathbb{C}(\xi\cdot s)] - \tfrac{1}{2}(\xi\cdot s)^T m(\xi\cdot s) + V. \tag{15}$$

Lemma 8. *If we assume that* $\xi \cdot s = \mathbb{M}^{-1}(\sigma - \mathbb{A}^T\mu_\perp)$ (*this is one of the relative equilibrium conditions in* Eq. (9)) *then the equation* $\xi \cdot \sigma = -\partial \bar{H}/\partial s$ *is equivalent to* $\mathrm{d}V_{\mu_\perp,\xi}/\mathrm{d}s = 0$.

Proof. Recalling the definitions $\mathbb{A} := \mathbb{I}_\mathrm{r}^{-1}\mathbb{C}$ and $\mathbb{M} := m - \mathbb{C}^T\mathbb{I}_\mathrm{r}^{-1}\mathbb{C}$, we have

$$V_{\mu_\perp,\xi} = \tfrac{1}{2}\mu_\perp^T\mathbb{I}_\mathrm{r}^{-1}\mu_\perp - \mu_\perp^T\mathbb{A}(\xi \cdot s) - \tfrac{1}{2}(\xi \cdot s)^T\mathbb{M}(\xi \cdot s) + V.$$

Hence

$$\begin{aligned}\frac{\mathrm{d}V_{\mu_\perp,\xi}}{\mathrm{d}s} &= \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{2}\mu_\perp^T\mathbb{I}_\mathrm{r}^{-1}\mu_\perp\right) - \mu_\perp^T\frac{\mathrm{d}\mathbb{A}}{\mathrm{d}s}(\xi \cdot s)\\ &\quad - \frac{1}{2}(\xi \cdot s)^T\frac{\mathrm{d}\mathbb{M}}{\mathrm{d}s}(\xi \cdot s) - (\mu_\perp^T\mathbb{A} + (\xi \cdot s)^T\mathbb{M})\frac{\mathrm{d}}{\mathrm{d}s}(\xi \cdot s) + \frac{\mathrm{d}V}{\mathrm{d}s}.\end{aligned}$$

Assuming $\xi \cdot s = \mathbb{M}^{-1}(\sigma - \mathbb{A}^T\mu_\perp)$, it follows that

$$\begin{aligned}\frac{\mathrm{d}V_{\mu_\perp,\xi}}{\mathrm{d}s} &= \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{2}\mu_\perp^T\mathbb{I}_\mathrm{r}^{-1}\mu_\perp\right) - \mu_\perp^T\frac{\mathrm{d}\mathbb{A}}{\mathrm{d}s}\mathbb{M}^{-1}(\sigma - \mathbb{A}^T\mu_\perp)\\ &\quad - \frac{1}{2}(\sigma - \mathbb{A}^T\mu_\perp)^T\mathbb{M}^{-1}\frac{\mathrm{d}\mathbb{M}}{\mathrm{d}s}\mathbb{M}^{-1}(\sigma - \mathbb{A}^T\mu_\perp) - \sigma^T\frac{\mathrm{d}}{\mathrm{d}s}(\xi \cdot s) + \frac{\mathrm{d}V}{\mathrm{d}s}.\end{aligned}$$

Note that differentiating the identity $\mathbb{M}\mathbb{M}^{-1} = \mathbb{I}$ gives

$$\mathbb{M}^{-1}\frac{\mathrm{d}\mathbb{M}}{\mathrm{d}s}\mathbb{M}^{-1} = -\frac{\mathrm{d}\mathbb{M}^{-1}}{\mathrm{d}s}.$$

Also, from the definition of the inverse dual action on S^* we have

$$\sigma^T\frac{\mathrm{d}}{\mathrm{d}s}(\xi \cdot s) = \frac{\mathrm{d}}{\mathrm{d}s}\sigma^T(\xi \cdot s) = -\frac{\mathrm{d}}{\mathrm{d}s}(\xi \cdot \sigma)^T s = \xi \cdot \sigma.$$

Hence

$$\begin{aligned}\frac{\mathrm{d}V_{\mu_\perp,\xi}}{\mathrm{d}s} &= \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{2}\mu_\perp^T\mathbb{I}_\mathrm{r}^{-1}\mu_\perp\right) - \mu_\perp^T\frac{\mathrm{d}\mathbb{A}}{\mathrm{d}s}\mathbb{M}^{-1}(\sigma - \mathbb{A}^T\mu_\perp)\\ &\quad + \frac{1}{2}(\sigma - \mathbb{A}^T\mu_\perp)^T\frac{\mathrm{d}\mathbb{M}^{-1}}{\mathrm{d}s}(\sigma - \mathbb{A}^T\mu_\perp) + \xi \cdot \sigma + \frac{\mathrm{d}V}{\mathrm{d}s} = \frac{\partial \bar{H}}{\partial s} + \xi \cdot \sigma. \quad \square\end{aligned}$$

Our calculations in this section imply that the general relative equilibrium conditions in Eq. (9) take the following form for simple mechanical systems,

$$\begin{cases} \mathbb{P}_{\mathfrak{k}}(\mathrm{Ad}_{g^{-1}}\eta) - \xi = 0 \\ \mathbb{P}_{\mathfrak{k}^\perp}(\mathrm{Ad}_{g^{-1}}\eta) = \dfrac{\partial \bar{H}}{\partial \mu_\perp} = \mathbb{I}_{\mathrm{r}}^{-1}\mu_\perp - \mathbb{A}\mathbb{M}^{-1}(\sigma - \mathbb{A}^T\mu_\perp) \\ -\mathrm{ad}^*_\xi \mu_\perp = \overline{\mathrm{ad}}^*_{\partial\bar{H}/\partial\mu_\perp}\mu_\perp + \mathrm{ad}^*_{\partial\bar{H}/\partial\mu_\perp} s \diamondsuit \sigma \\ \xi \cdot s = \dfrac{\partial \bar{H}}{\partial \sigma} = \mathbb{M}^{-1}(\sigma - \mathbb{A}^T\mu_\perp) \\ \dfrac{\mathrm{d}V_{\mu_\perp,\xi}}{\mathrm{d}s} = 0. \end{cases} \tag{16}$$

Remark 9. Note that the second and fourth equations may be jointly expressed as

$$(\mathbb{P}_{\mathfrak{k}^\perp}(\mathrm{Ad}_{g^{-1}}\eta), \xi \cdot s) = (\mathbb{F}\bar{L})^{-1}(\mu_\perp, \sigma)$$

In summary, we have the following,

Proposition 10. *Let* $(g, \mu_\perp, s, \sigma) \in G \times \mathfrak{k}^\circ \times T^*S$ *and* $z = \rho \circ \varphi(g, \mu_\perp, s, \sigma)$. *Then* z *is a relative equilibrium of* X_H *with velocity* $\eta \in \mathfrak{g}$ *if and only there exists a* $\xi \in \mathfrak{k}$ *such that the following conditions are satisfied,*

(1) $\mu_\perp, \xi$ *and* s *are such that* s *is a critical point of* $V_{\mu_\perp,\xi}$ *and*

$$-\mathrm{ad}^*_\xi \mu_\perp = \overline{\mathrm{ad}}^*_{\mathbb{I}_{\mathrm{r}}^{-1}\mu_\perp - \mathbb{A}(\xi\cdot s)}\mu_\perp + \mathrm{ad}^*_{\mathbb{I}_{\mathrm{r}}^{-1}\mu_\perp - \mathbb{A}(\xi\cdot s)}(s \diamondsuit [\mathbb{M}(\xi \cdot s) + \mathbb{A}^T\mu_\perp])$$

(2) $\sigma = \mathbb{M}\,(\xi \cdot s) + \mathbb{A}^T\mu_\perp$
(3) $\eta = \mathrm{Ad}_g(\xi + \mathbb{I}_{\mathrm{r}}^{-1}\mu_\perp - \mathbb{A}\mathbb{M}^{-1}(\sigma - \mathbb{A}^T\mu_\perp))$.

7. Comments

We have outlined a framework for studying mechanical systems determined by a symmetric Lagrangian on TQ or a symmetric Hamiltonian on T^*Q, at configurations near a given one, q_0, with nontrivial isotropy. We have found tubes around "$(q_0, 0)$", meaning $0 \in T_{q_0}Q$ or $0 \in T^*_{q_0}Q$, which we consider as twisted parametrisations for TQ and T^*Q. These parametrisations are semi-local in the configuration space but global in the fibre direction. This means that we can study all local dynamics near a given configuration point using one parametrisation. In particular, we can study all relative equilibria with configurations near q_0. This is particularly valuable given the incomplete development of slice theorems and reduction theory for lifted actions. Indeed, constructive slice coordinates based at a general relative equilibrium (q_0, p_0) are not yet available, but we can study these relative equilibria using our slice coordinates based at $(q_0, 0)$; it is not necessary for $(q_0, 0)$ to be an equilibrium or relative equilibrium.

We have defined a Legendre transform using our twisted parametrisations of TQ and T^*Q. This is valuable because in many mechanical systems are most naturally described with a Lagrangian, whereas relative equilibria are most naturally described in a Hamiltonian setting. Beginning with a given Lagrangian on TQ we are able to state necessary and sufficient conditions for the existence of relative equilibria, using slice coordinates on T^*Q. Slice coordinates have the advantage that a single coordinate system covers configurations of different isotropy types. In the case of simple mechanical systems, we have stated the relative equilibria conditions in terms of an *augmented-amended potential* which generalises both the amended and the augmented potentials familiar from the case of free actions.

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References

[1] R. Abraham, J.E. Marsden, Foundations of Mechanics, second ed., Addison-Wesley, 1978.
[2] L. Bates, E. Lerman, Proper group actions and symplectic stratified spaces, Pacific J. Math. 181 (2) (1997) 201–229.
[3] G.E. Bredon, Introduction to Compact Transformation Groups, Academic Press, 1972.
[4] R. Cushman, L. Bates, Global Aspects of Classical Integrable Systems, Birkhäuser, 1997.
[5] R. Cushman, J. Sniatycki, Differential structure of orbit spaces, Canad. J. Math. 53 (4) (2001) 715–755.
[6] V. Guillemin, S. Sternberg, A normal form for the moment map, in: S. Sternberg (Ed.), Differential Geometric Methods in Mathematical Physics, Mathematical Physics Studies, vol. 6, D. Reidel, 1984.
[7] J.J. Duistermaat, J.A.C. Kolk, Lie Groups, Springer-Verlag, 1999.
[9] I.N. Kozin, R.M. Roberts, J. Tennyson, Relative equilibria of D_2H^+ and H_2D^+, Mol. Phys. 98 (5) (2000) 295–307.
[10] E. Lerman, S.F. Singer, Stability and persistence of relative equilibria at singular values of the moment map, Nonlinearity 11 (1998) 1637–1649.
[11] D. Lewis, Lagrangian block diagonalization, J. Dyn. Differential Equations 4 (1) (1992).
[12] C.-M. Marle, Modèle d'action hamiltonienne d'un groupe de Lie sur une variété symplectique, vol. 43 (2), Rendiconti del Seminario Matematico, Università e Politecnico, Torino, 1985, pp. 227–251.
[13] J.E. Marsden, Lectures on Mechanics, London Mathematical Society, Lecture Note Series, Cambridge University Press, 1992.
[14] J.E. Marsden, T.S. Ratiu, Introduction to Mechanics and Symmetry, second ed., Springer, 1999.
[15] R. Montgomery, J.E. Marsden, T.S. Ratiu, Gauged Lie-Poisson structures, Cont. Math. AMS 128 (1984) 101–114.
[16] J.E. Marsden, M. Perlmutter, The orbit bundle picture of cotangent bundle reduction, C. R. Math. Acad. Sci. Soc. R. Can. 22 (2) (2000) 35–54.
[19] J.E. Marsden, A. Weinstein, Reduction of symplectic manifolds with symmetry, Rep. Math. Phys. 5 (1974) 121–130.
[20] J.-P. Ortega, Symmetry, Reduction, and Stability in Hamiltonian Systems, Ph.D. thesis, University of California, Santa Cruz, 1998.
[22] J.-P. Ortega, T.S. Ratiu, Stability of Hamiltonian relative equilibria, Nonlinearity 12 (3) (1999) 693–720.
[25] J.-P. Ortega, T.S. Ratiu, A symplectic slice theorem, Lett. Math. Phys. 59 (1) (2002) 81–93.
[26] J.-P. Ortega, T.S. Ratiu, Momentum maps and Hamiltonian reduction, Progress in Mathematics, Birkhäuser, 2004.

[27] G.W. Patrick, Relative equilibria in Hamiltonian systems: the dynamic interpretation of nonlinear stability on a reduced phase space, J. Geom. Phys. 9 (2) (1992) 111–119.
[28] R. Palais, On the existence of slices for actions of non-compact Lie groups, Ann. Math. 73 (1961) 295–323.
[29] M. Perlmutter, M. Rodrguez-Olmos, M.E. Sousa-Dias, Singular cotangent bundle reduction, in preparation.
[31] M. Roberts, C. Wulff, J.S.W. Lamb, Hamiltonian systems near relative equilibria, J. Diff. Eq. (2002), in press.
[32] T. Schmah, A cotangent bundle slice theorem, 2004, preprint.
[33] R. Sjamaar, E. Lerman, Stratified symplectic spaces and reduction, Ann. of Math. 134 (1991) 375–422.
[34] J.K.G. Watson, Mol. Phys. 19 (1970) 465.